A comprehensive method of analysis of case furniture

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Abstract

Finite element methods of analysis were used to investigate the deflection characteristics of furniture cases. Results of the analyses indicated that the deflection of shelves is identical to that of the top and bottom and the deflection of partitions is equivalent to that of the sides in a case. An exact equation was developed which predicts the deflection of an unsupported corner of a case which may contain both shelves and partitions. Furniture manufacturers may use this equation to optimize paneled case designs.

The rational design of case furniture requires that methods of analysis be available which can be used to determine the deflections and stiffness of the case along with the forces which operate at the various joints. The methods required for such analyses have been slow to evolve, however. Presumably, the first published analysis of a case was that carried out by Kotas (9, 10) on an open-faced five-sided box. Results of this research were later incorporated into a small design manual (11).

Eckelman (1, 4) subsequently developed a method of analysis for cases based on the inter-related deflections of the various corners and the stiffnesses of the individual panels. Ganowicz and Rogozinski (8) applied the principles of internal work to the analysis of case furniture, and Ganowicz et al. (6) and Ganowicz and Kwiatkowski (7) subsequently developed this analysis further and evaluated the forces acting at the corners of a case under two loading conditions. Eckelman and Resheidat (5) subsequently elaborated on this work.

All of the analyses cited above deal with simple five-sided cases. None treat the more general problem of analyzing cases with shelves and partitions. Although this problem can be treated through an extension of the methods utilized by Ganowicz et al. (6), it was felt worthwhile to attempt the development of such theory through the use of existing computer based methods of analysis since the use of such techniques greatly facilitates the execution of the tedious calculations normally required. Additionally, such a line of development was considered useful since this would facilitate analysis of more complex plate-on-frame construction which cannot be treated by means of existing simple plate theory alone.

Background

In previous works (1, 4), it was shown that the deflection of one corner of a five-sided case (Fig. 1) could be expressed by means of the expression

$$Y = \frac{F}{\frac{f_{A}}{y_{4}}(\text{top}) + \frac{f_{L}}{y_{2}}(\text{bot}) + \frac{b^{2}f_{L}}{a^{2}y_{1,3}}(\text{sides}) + \frac{a^{2}f_{L}}{c^{2}y_{0}}(\text{rear})}$$

[1]

where \(f_{L}/y_{4}\) refers to the stiffnesses of the appropriate panels comprising the case; \(Y\) refers to the deflection of a free corner of the case; \(F\) refers to a vertical load applied at the free corner; \(a\) refers to the width of the case, \(b\) refers to the height, and \(c\) refers to the depth of the case (inches). The stiffness of the panels may be measured experimentally by supporting the panel at three corners, applying a vertical load, \(f\), to the free corner, and measuring the resulting deflection, \(y\), of the corner (Fig. 2). Alternatively, when solid homogeneous panels are used, the stiffness may be calculated by means of the expression

$$f/y = t^{2}G/(3L_{1}L_{2})$$

[2]

where \(t\) refers to the thickness of a panel (inches); \(G\) refers to the modulus of rigidity of the panel (psi); \(L_{1}\) refers to the length of the panel about the axis of twist (inches); and \(L_{2}\) refers to the width of the panel (14).

The deflection of each corner of the case is specifically related to every other corner, and in particular,
to the deflection of the free corner of the case. Thus, Figure 1,

\begin{align*}
\gamma_{23} &= \gamma_{53} = Y \\
\gamma_{52} &= \gamma_{62} = (b/a) \cdot Y \\
\gamma_{31} &= \gamma_{41} = (b/c) \cdot Y
\end{align*}

where \( \gamma_{23} \) refers to the vertical deflection of corner 2 in the \( X_3 \)-direction, \( \gamma_{52} \) refers to the deflection of corner 5 in the \( X_2 \)-direction, \( \gamma_{31} \) refers to the deflection of corner 8 in the \( X_1 \)-direction, etc.

The forces acting perpendicular to the faces of the panels at each corner are related to these deflections by means of Equation [2]; i.e.,

\[ f_i = \left( \rho_1 G_y \left( 3 - L_{11} L_{21} \right) \right) y_i \]  

where the subscript \( i \) refers to the appropriate panel.

Intuitively, it appears that the contributions of shelves and partitions in a case could be treated in the same manner as the bottoms and sides, i.e., the shelves should twist in a fixed relationship to the bottom and top, and the partitions should twist in a definite relationship to the sides. Thus, the stiffness of a complete case, presumably, could be calculated by adding the appropriate terms to the denominator of Equation [1] to account for the added stiffness provided by these members.

In order to examine these relationships, a frame plate element was devised which could be used with computer based matrix methods of analysis to simulate the structural behavior of solid panels in case furniture. As stated previously, development of such an element is of particular importance since it also offers the possibility of providing a means of analyzing complex plate-on-frame construction as well as pure plate construction and thereby furthering the development of case analysis theory.

Important considerations in development of the frame type model for plates were as follows: 1) the frame element had to be highly resistant to deformation in its own plane, and 2) the frame element had to possess the same torsional resistance as the panel it replaced. Several frame type elements could undoubtedly be devised which would satisfy these requirements, but an element which appeared to be particularly suited for purposes of analyses was a four member frame with cross bracing (Fig. 3). In this element, the cross bracing provides the torsional strength required, whereas the outer perimeter members possess high axial stiffness and serve to complete the outer framework. Importantly, the ends of all the members which frame into the four perimeter corner joints must be quite flexible to prevent the outer frame members from contributing to the torsional stiffness of the frame.

A decided advantage of the frame type model is that the size and properties of the various members can be readily determined. In practice, the deflection of the free corner of a true panel is first determined under the action of a vertical unit load by means of Equation [2]. Next, either the cross section or the modulus of elasticity (or both) of the cross members in the frame plate element is adjusted to give an identical free corner deflection. The deflection of the free corner of the frame plate element model is given by the expression

\[ y = \frac{8FL^3}{(48EI)} \]  

where \( F \) refers to the unit vertical load; \( L \) is the length of the cross member (inches); \( E \) is the modulus of elasticity of the cross members (psi); and \( I \) is the moment of inertia of the cross members (in.\(^4\)). This equation simply expresses the relationship that the deflection of the free end of a crossed beam is equal to eight times the mid-span deflection of a simply supported beam. Rearranging terms gives

\[ EI = \frac{8FL^3}{(48y)} \]

so that \( E \) or \( I \) can be readily determined. If members are used which are 1 inch square, this expression reduces to the form

\[ E = 2L^3/y \]
for the case of a unit load so that the modulus of elasticity of the cross members may be readily calculated.

The purpose of the outside members is to help prevent in-plane distortion of the frame, i.e., to hold the corners at fixed distances apart. They must, therefore, have high resistance to axial deformation. At the same time, however, they must not develop bending forces. Ideally, therefore, the ends of these members should be pinned. Alternatively, in systems where this cannot be done, the diameters of the members may be made quite small, say 0.1 to 0.25 inch, to ensure that they cannot develop significant bending forces; but their modulus of elasticity is set at a high value, say 3.0 \times 10^6 psi, to minimize axial deformation under load. Finally, in those methods of analysis which will allow the use of semi-rigid joints (2, 12, 13), the stiffness values, or, Z-values as they are called, for these corner connections can be set equal to 1.0 since use of Z-values of this magnitude ensures that the bending forces developed by the members will be negligible. When this can be done, the diameter of the outside frame members should be increased to say 1.0 inch to further reduce axial deformations.

It is also important that the cross members not develop significant bending forces at the perimeter corner joints. In the case of those programs which allow only semi-rigid joints to be used, the joints at each end of the cross members may be treated as above, i.e., the Z-values may be set equal to 1. In methods of analysis which do not allow semi-rigid joints, short flexible members may be introduced at the ends of the cross braced members which will produce the same effect. This method introduces 24 additional unknowns into each plate element, however. The joint formed where the members cross must be rigid, however, since the intent is to create a plate element in which all of the members are connected to the corner joints by means of flexible joints while the two crossed elements are connected at midspan by means of a fixed joint.

**Evaluation of single plate element**

An analysis was carried out in which the deflection characteristics of a frame plate element were compared with the deflections predicted by Equation [2] for a corresponding solid panel. The deflection of a solid panel supported at three corners and loaded at the fourth as shown in Figure 2 was first determined. Characteristics of the panel were fixed as follows: modulus of rigidity = 200,000 psi; thickness = 1 inch; length = 50 inches; width = 25 inches. Deflection of the free corner accord-

ing to panel theory, Equation [2], amounts to 

\[
\begin{align*}
y &= 1 lb \cdot 3.25\ in \cdot 50\ in \cdot 200,000\ psi \cdot 1\ in^3 \\
  &= 0.01875\ inches
\end{align*}
\]

The modulus of elasticity of a cross member of the frame plate element used to represent this plate (Fig. 3) may be found through the use of Equation [6]. If the cross members are assumed to be 1 inch square, the modulus of elasticity amounts to 

\[
E = 2.0(55.9)^{2}/0.01875 = 1.8634 \times 10^7\ psi
\]

where the length of the cross members is 55.9 inches. The diameters of the perimeter elements of this frame model were set equal to 1.0 inch with a modulus of rigidity of 3.0 \times 10^6 psi.

A structural analysis of this frame plate element was carried out with CODOFF (3). Z-values for the joints were set equal to 1.0. Calculated deflection of the free corner amounted to 0.01875 inch. This result indicated that when the cross sections and moduli of elasticity are determined as described above, the frame plate element may be expected to give predictions of free corner deflections essentially identical to those predicted by Equation [2].

**Evaluation of basic five-sided case**

The deflection characteristics of a simple five-sided case without shelves or partitions (Fig. 4a) were next considered. The case was considered to be constructed of 1-inch-thick particleboard with a modulus of rigidity of 200,000 psi. Dimensions of the case were 50 inches wide by 25 inches deep by 100 inches high (Fig. 1).

**Solid panel solution**

The deflection characteristics of the case and the internal forces acting at the corners were analyzed by means of Equation [1]. Deflection of the free corner, \(Y\), was found to be 

\[
Y = 1 \int \frac{1-G}{3\cdot25-50} + \frac{1-G}{3\cdot25-50} \\
+ \frac{2\cdot(100)^2}{25\cdot25-100} + \frac{1-G}{50^2-25-50} + \frac{1-G}{25\cdot25-100}
\]

or, 0.001875 inch under the action of a 1-pound vertical load where \(G = 200,000\ psi\).

Deflection of corner 5 of this case, (Fig. 4a) according to plate theory amounts to \((b/a) = 0.001875\), or, 0.00375 inch in the \(X_2\)-direction; similarly, deflection of corner 6 in the \(X_2\)-direction amounts to \((b/c) = 0.001875\), or, 0.0075 inch.

Forces acting at the corners of the panels perpendicular to the face of the panels are found from Equation [3]. Thus, the absolute value of the forces acting at corners 5, 6, 7, 8 of the top panel was found to be 

\[
f = (1-G)(0.001875)/(3.0\cdot25-50) = 0.1\ pound
\]

Similarly, the absolute value of the forces acting at corners 1, 2, 3, 5, 7 of the left and corners 2, 4, 5, 7 of the right side were found to be 

\[
f = (1-G)(0.00375)/(3.0\cdot25-100) = 0.1\ pound
\]

Likewise, the absolute value of the forces acting at corners 3, 4, 8, 9 of the rear panel was found to be 

\[
f = (1-G)(0.0075)/(3.0\cdot50-100) = 0.1\ pound
\]
Frame plate element solution

The frame plate element case model which simulates the solid panel model just described is shown in Figure 4b. Required properties of the cross bars were found by solving for $EI$ values such that the deflection of the free corner of the frame plate element under the action of a unit load, Equation [5], is equal to that of the free corner of the panel the frame plate element replaces, Equation [2]. Setting these expressions equal to one another and solving for $EI$ gives

$$EI = (t^3 G L_0^3)/(18 L_1 L_2),$$

where $L_0$ refers to the length of the cross bar. If the cross sections of the cross bars are set equal to 1 inch square, the equation reduces to the form

$$E = 2 t^3 G L_0^3/(3 L_1 L_2).$$

For the plate element shown, all of the cross members were set equal to 1 inch square; diameter of the perimeter members was 1.0 inch. MOE of the perimeter members was $3.0 \times 10^6$ psi. The MOE of the cross members of the bottom and top frames was set equal to $1.8634 \times 10^7$, the MOE of the sides to $3.7270 \times 10^7$, and the MOE of the back to $5.8415 \times 10^7$ psi.

Under the action of a 1-pound vertical load applied at joint 2, the deflection at this point as determined by computer analysis (3) amounted to 0.001876 inch. Deflection of joint 9 in the negative $X_2$-direction amounted to 0.003749 inch; similarly, deflection of joint 10 amounted to 0.007499 inch. The forces which act perpendicular to the ends of the cross members as determined from the computer analysis were as follows: for the left side, 0.09998 pound; for the right side, 0.09998 pound; for the top, 0.1 pound; and for the bottom 0.1000 pound. These values are to be compared to values of 0.1 pound for all of these points as calculated above with the solid panel model.

These results provided convincing evidence that the frame plate element could be expected to produce essentially the same results as those obtained with solid panel deflection analysis. Close results were obtained in additional analyses also. The thickness of the bottom was doubled in one analysis, for example. Solid panel theory predicted a free corner deflection of 0.001103 inch; the frame plate analysis indicated a deflection of 0.001104 inch.

In summary, the results of several analyses carried out with the frame plate model produced estimates of deflections and forces which for all practical purposes were identical to those produced with analyses carried out with the solid panel model. Such slight differences as noted above occur because deformation of members cannot be completely eliminated and joints are not truly pinned — not because of conceptual problems in the element itself.

Evaluation of five-sided case with shelf

A case was next considered which contained a shelf. This case was identical to the basic five-sided case described in the previous section but had a shelf inserted at midheight. The joint and member numbering systems for this case are shown in Figure 5. Owing to symmetry, the dimensions of the bottom, top, shelf, and side panels were identical. The moduli of elasticity of the cross members of these panels, accordingly, were also identical, i.e., MOE = $1.8634 \times 10^7$ psi; MOE of the back panel cross members was $1.8856 \times 10^7$. 
the deflection of shelves is identical to the deflection of the tops and bottoms, then the deflection of the free corner of the basic five-sided case without a shelf but with the stiffness of the bottom doubled should be the same as that of the case with shelf (Fig. 5). Deflection of the free corner of the case with bottom stiffness doubled amounted to 0.001705 inch. The deflection of the frame plate model case was then determined when the stiffnesses of the cross members and the corresponding joints of the shelf element were set equal to those of the bottom. Deflection of the free corner was found to be 0.001705 inch. Thus, it was concluded that the structural behavior of the shelf was, in fact, identical to that of top and bottom and could be treated as such in structural analyses of cases constructed of panels.

Additional analyses provided consistent support for this concept. In particular, an analysis of a case containing several shelves indicated that all of the shelves deflected identically to the top and bottom. This result indicated that shelf deflection is independent of position in the case.

**Evaluation of case with shelf and partition**

Intuitively, it can be seen that relationships similar to those for shelves also hold for partitions except that a partition must be treated as a side. To evaluate this relationship, a case was next considered which was identical in size to those previously analyzed but which contained both a shelf and a partition. These were located at the midpoints of the case (Fig. 6). An analysis was first carried out with the stiffness of the partition and the shelf set essentially equal to zero in order to determine if the basic subdivided frame plate model case had the same deflection characteristics as the simple five-panel case previously analyzed. Deflection of the free corner of the case under these conditions was found to be 0.001877 inch (versus 0.001875 for the solid panel case). Importantly, the torsional rotation of the sides and partition were the same, and, as expected from the preceding analysis, the torsional rotation of the top, shelf, and bottom were found to be identical. This analysis indicated that the frame plate model constructed of half sides, tops, bottoms and shelves, and quarter backs gave results essentially identical to those obtained with the comparable solid plate model (Fig. 4a).

An analysis was next carried out on this case after the stiffness of the shelf had been set equal to that of the bottom and the stiffness of the partition to that of the sides. Deflection of the free corner of the case under these conditions amounted to 0.001445 inch. For purposes of comparison, an analysis was then carried out in which the stiffnesses of the shelf and partition were set equal to essentially zero while the stiffnesses of the bottom and the left side of the case were doubled. Deflection of the free corner of the case under these conditions amounted to 0.001446 inch (versus 0.001445 in the previous analysis). The close agreement of these values provided additional confidence in the ability of the frame plate element model to provide consistent results. Finally, an analysis was carried out of the basic five-sided solid panel case by means of Equation [1] in which the stiffnesses of both the bottom and the one side were also doubled. Results of this analysis gave a deflection of...
0.001442 inch (versus the value of 0.001446 obtained above with the frame plate model). The close agreement of this deflection with that obtained in the previous frame plate analysis again indicated the ability of the frame plate model to accurately predict deflections under a variety of conditions.

Development of general method of analysis

Results of the previous analyses demonstrated that the deflection characteristics of a case with shelves or with shelves and partitions are both regular and unique. Specifically, these analyses showed that the torsional displacements of shelves are identical to those of the top and bottom, and likewise, the torsional displacements of interior partitions are identical to those of the sides. It follows, therefore, that the contribution of these components to the stiffness of a case as a whole can be determined by adding appropriate terms to the denominator of Equation [1] to account for these elements.

As an example, the deflection of the free corner of the frame plate case with shelf was found to be 0.001705 inch as was that of the solid panel case in which the stiffness of the bottom was doubled. This deflection can also be found, as stated above, by inserting an additional term into Equation [1] to account for the shelf. Since the characteristics of the top, bottom, and shelf are identical, the terms for the top and bottom in Equation [1] may be replaced by a single term multiplied by a factor of three as shown below.

\[
Y = \frac{1}{3} \cdot \frac{1 \cdot G}{25 \cdot 50} + 2 \cdot \frac{(100)^2}{(50)^2} \cdot \frac{1 \cdot G}{3 \cdot 25 \cdot 100} + \frac{(100)^2}{(25)^2} \cdot \frac{1 \cdot G}{3 \cdot 50 \cdot 100}
\]

or, \( Y = 0.001705 \) inch. As can be seen, both methods yield the same result. Similarly, if this operation is
carried out for the case containing both a partition and a shelf, the following expression results where the first term in the denominator refers to the top, bottom, and shelf, and the second term refers to the left and right sides and the interior partition.

\[
Y = \frac{1}{3} - \frac{1\cdot G}{3\cdot 25\cdot 50} + \frac{3\cdot (100)^2}{(50)^2} \cdot \frac{1\cdot G}{3\cdot 25\cdot 100} + \frac{(100)^2}{(25)^2} \cdot \frac{1\cdot G}{3\cdot 50\cdot 100}
\]

or, \( Y = 0.001446 \) inch. This result should be compared to the identical value of 0.001446 inch obtained in the frame plate element model analysis.

These results may be generalized and put in the form of the following equation.

\[
Y = F \sum \frac{f}{y} \left( \frac{f}{(bot)} + \frac{f}{(top)} + \frac{f}{(shelves)} \alpha^2 \right) + 2 \cdot \frac{b^2 f}{a^2 y} + \frac{b^2}{a^2} \frac{f}{(part)} \beta^2 + \frac{b^2 f}{c^2 y} \frac{f}{(back)}
\]

where \( \alpha \) is the ratio of the length of the shelf to the length of the bottom, and \( \beta \) is the ratio of the height of the partitions to the height of the sides. Inclusion of the \( \alpha \) and \( \beta \) terms allows this expression to be used with partial shelves which frame into partitions and with partial partitions which frame into shelves.

### Discussion

Results of this study indicate that the structural behavior of a case constructed of panels may be accurately determined by means of computer based methods of analysis developed for frame-type structures provided that the panels are replaced by appropriate frame plate elements. In particular, it was found that a frame plate type element with cross bars in which all of the members were attached to the perimeter corner joints by means of pinned joints but the central cross joint was of rigid construction provided an excellent model of a panel with respect to structural behavior. Development of this model was important both because it provided insight into the structural behavior of solid panel cases and also because it provides a means for the subsequent treatment of panel on frame construction.

Of particular interest, development of the model and the subsequent frame plate analyses that were carried out indicated that the shelves in a case deflect identically to the tops and bottoms while the partitions deflect identically to the sides. This result allowed an exceptionally simple design expression to be formulated for calculating the deflection of a five-sided case containing any number of shelves and partitions.

Results of this study also indicate that methods of analysis based on plate theory for open faced five-sided cases can readily be extended to treat cases with any number of shelves and partitions. This follows because the shelves in a case deflect identically to the top and bottom, and partitions deflect identically to the sides.

### Literature cited

5. and M. Reshehidat. 1983. The analysis of five-side furniture cases. (In progress.)